UDC 62-50

ON A CLASS OF SINGULARLY-PERTURBED PROBLEMS OF OPTIMAL CONTROL

PMM Vol. 42, № 2, 1978. pp. 228-232

M. G. DMITRIEV
(Krasnoiarsk)

(Received August 13, 1976)

It is shown that the optimal trajectories in the linear problem of optimal control with a quadratic functional and a large parameter accompanying the control are close, in a certain sense, to the solutions of a specially constructed problem of the same type but of lower dimension. The controls in this new problem are the "rapid" components of the phase variables.

In the applications, the mathematical model of the controlled object may contain large gain coefficients of the order of ε^{-1} ($\varepsilon > 0$ is a small parameter). Such models are singularly-perturbed [1], some of their motions are rapid, and some are slow. This was pointed out, in particular, by Gerashchenko in [2] where the problem of separating the motions in a closed control system with a large gain coefficient accompanying the control was discussed.

Problems of optimal control with differential constraints of this type are usually solved in the following manner: equations containing ε^{-1} are multiplied by ε, and then assumed $\varepsilon = 0$ whereupon the whole control vector or some of its coordinates vanish and the corresponding phase variables become the controls in the resulting problem of lower dimension. For example, the computations in the problem of controlling the flight of a rocket in vacuo (plane case) which was shown to the author by N. N. Moiseev, are carried out with the rotation about the center of mass neglected, i. e. the phase variable represented by the angle related to the direction of thrust is regarded as the control, and the control consisting of the torsional moment is neglected (see also [3]).

The above approach is substantiated using, as an example, the problem of analytic construction of a regulator of state.

We require to find an r-dimensional continuous function $v(t)$ minimizing the functional

$$I(v) = \frac{1}{2} x'(T) F x(T) + \frac{1}{2} \int_{t_0}^{T} (x' Q(t) x + v' R(t) v)\, dt \tag{1}$$

on the trajectories of the system

$$\begin{aligned} x_1^{\cdot} &= A_1(t)\, x_1 + A_2(t)\, x_2 + B_1(t)\, v, \quad x_1(t_0) = x_1^{\circ} \\ x_2^{\cdot} &= A_3(t)\, x_1 + A_4(t)\, x_2 + \varepsilon^{-1} B_2(t)\, v, \quad x_2(t_0) = x_2^{\circ} \end{aligned} \tag{2}$$

$$x = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \quad x_1 \in E^n, \quad x_2 \in E^m, \quad Q = \begin{Vmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{Vmatrix}, \quad F = \begin{Vmatrix} F_1 & F_2 \\ F_2' & F_3 \end{Vmatrix}$$

Here all matrices are twice continuously differentiable in t, $R(t)$ and $Q(t)$ are matrices positive-definite in $[t_0, T]$, F is a constant positive-definite matrix, the prime denotes transposition.

We shall assume for simplicity that $B_1(t) \equiv 0$. Let $B_2(t)$ be an $m \times r$ matrix of rank r, and $m = r$. In this case we can write $B_2(t) \equiv E$ (E is a unit matrix of dimension r) without loss of generality. Taking these assumptions into account, we can rewrite (2) in the form

$$\begin{aligned} x_1^{\cdot} &= A_1(t)x_1 + A_2(t)x_2, \quad x_1(t_0) = x_1^{\circ} \\ \varepsilon x_2^{\cdot} &= \varepsilon A_3(t)x_1 + \varepsilon A_4(t)x_2 + v, \quad x_2(t_0) = x_2^{\circ} \end{aligned} \tag{3}$$

i.e. the system contains a small parameter ε accompanying the derivatives, and it can be expected that the boundary value problem of the Pontriagin maximum principle for the problem (1), (3) will be singularly perturbed. Indeed we have $(v = R^{-1}p_2)$

$$\begin{aligned} & x_1^{\cdot} = A_1(t)x_1 + A_2(t)x_2 \\ & \varepsilon x_2^{\cdot} = \varepsilon A_3(t)x_1 + \varepsilon A_4(t)x_2 + R^{-1}p_2 \\ & p_1^{\cdot} = Q_1(t)x_1 + Q_2(t)x_2 - A_1'(t)p_1 - \varepsilon A_3'(t)p_2 \\ & \varepsilon p_2^{\cdot} = Q_2'(t)x_1 + Q_3(t)x_2 - A_2'(t)p_1 - \varepsilon A_4'(t)p_2 \\ & x_1(t_0) = x_1^{\circ},\ x_2(t_0) = x_2^{\circ};\ p_1(T) = -(F_1x_1(T) + F_2x_2(T)) \\ & p_2(T) = -\varepsilon^{-1}(F_2'x_1(T) + F_3x_2(T)) \end{aligned} \tag{4}$$

Here $p_1(t)$, $p_2^{\circ}(t) = \varepsilon p_2(t)$ denote the conjugate variables of the maximum principle, $x_2(t)$ and $p_2(t)$ are the rapid variables while $x_1(t)$ and $p_1(t)$ are the slow variables. When $\varepsilon = 0$, the additional conditions for x_2 and p_2 at $t = t_0$ and $t = T$ respectively are, in general, lost, and this leads to the appearance of boundary layer zones near these points.

The singular character of the condition for $p_2(T)$ when $\varepsilon = 0$, constitutes the characteristic feature of (4). As the result of this, the boundary condition for the slow conjugated variable changes discontinuously in the limit as $\varepsilon \to 0$. This implies that the term of the limiting variational problem of reduced dimensionality appearing outside the integral also changes discontinuously. Problems of this type have been studied earlier (*).

Using the results of [7], we shall consider the passage to the limit in the solutions of (4), passing from (4) to the Cauchy problem for the Riccati equation

$$K^{\cdot} = -KA - A'K + KBR^{-1}B'K - Q, \quad K(T) = F \tag{5}$$

$$A = \begin{Vmatrix} A_1 & A_2 \\ A_3 & A_4 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 \\ E/\varepsilon \end{Vmatrix}, \quad K = \begin{Vmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_2' & \varepsilon K_3 \end{Vmatrix}$$

We note that the passage from (4) to (5) utilizes the relation

$$\begin{Vmatrix} p_1 \\ \varepsilon p_2 \end{Vmatrix} = K \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} \tag{6}$$

*) Dmitriev M.G. Study of the singular perturbations of the optimal control problems. Candidate's Dissertation. Dnepropetrovsk, 1972. See also [5 — 7].

Let $x_1(t,\varepsilon)$, $x_2(t,\varepsilon)$, $v(t,\varepsilon)$, $I_\varepsilon{}^*$ denote, in the problem (1), (3), the optimal trajectories, the control and the minimum value of the functional, respectively.

Theorem. When $\varepsilon \to 0$ $x_1(t,\varepsilon) \to \bar{x}_1(t)$ uniformly in $t \in [t_0, T_1] \subset [t_0, T]$, $x_2(t,\varepsilon) \to \bar{x}_2(t)$ uniformly in $t \in [T_0, T_1] \subset [t_0, T]$, $v(t,\varepsilon) \to 0$ uniformly in $t \in [T_0, T_1]$ and $I_\varepsilon{}^* \to \bar{I}^*$. Here $\bar{x}_1(t)$, $\bar{x}_2(t)$ and $\bar{I}^*$ denote, respectively, the optimal trajectory, optimal control and minimum value of the functional of the problem

$$\bar{I}(\bar{x}_2) = \frac{1}{2}\bar{x}_1'(T)(F_1 - F_2F_3^{-1}F_2')\bar{x}_1(T) + \frac{1}{2}\int_{t_0}^{T} \begin{Vmatrix}\bar{x}_1\\ \bar{x}_2\end{Vmatrix} Q(t) \begin{Vmatrix}\bar{x}_1\\ \bar{x}_2\end{Vmatrix} dt \tag{7}$$

$$\bar{x}_1^{\bullet} = A_1(t)\bar{x}_1 + A_2(t)\bar{x}_2, \qquad \bar{x}_1(t_0) = x_1{}^0$$

Proof. Let us write (5) block by block

$$\begin{aligned}
\frac{dK_1}{dt} &= -K_1A_1 - A_1K_1 - \varepsilon K_2A_3 - \varepsilon A_3'K_2' + K_2R^{-1}K_2' - Q_1 \\
\varepsilon\frac{dK_2}{dt} &= -K_1A_2 - \varepsilon K_2A_4 - \varepsilon A_1'K_2 - \varepsilon A_3'K_3 + K_2R^{-1}K_3 - Q_2 \\
\varepsilon\frac{dK_3}{dt} &= -\varepsilon K_2'A_2 - \varepsilon A_2'K_2 - \varepsilon K_3A_4 - \varepsilon A_4'K_3 + K_3R^{-1}K_3 - Q_3 \\
K_1(T) &= F_1,\ K_2(T) = F_2/\varepsilon,\quad K_3(T) = F_3/\varepsilon
\end{aligned} \tag{8}$$

The conditions imposed on the coefficients of (2) and (3) imply that the assumptions of the corollary to the theorem in [7] hold, i.e. when $\varepsilon \to 0$, the solutions $K_i(t,\varepsilon)$, $i = 1, 2, 3$, of the problem (8) tend to the solutions $\bar{K}_i(t)$, $i = 1, 2, 3$ of the problem obtained from (8) by setting $\varepsilon = 0$ (where $\bar{K}_1(T) = F_1 - F_2F_3{}^{-1}F_2'$) uniformly in t on every segment $[t_0, T_1] \subset [t_0, T]$. Finding the values of $\bar{K}_2$ and $\bar{K}_3$ and substituting them into equations for $\bar{K}_1$, we obtain

$$\begin{aligned}
\frac{d\bar{K}_1}{dt} = {} & -\bar{K}_1(A_1 - A_2Q_3^{-1}Q_2') - (A_1' - Q_2Q_3^{-1}A_2')\bar{K}_1 + \\
& \bar{K}_1A_2Q_3^{-1}A_2'\bar{K}_1 - (Q_1 - Q_2Q_3^{-1}Q_2'), \qquad \bar{K}_1(T) = F_1 - F_2F_3^{-1}F_2'
\end{aligned} \tag{9}$$

The positive-definiteness of the matrices Q and F implies the positive-definiteness of the matrices $Q_1 - Q_2Q_3{}^{-1}Q_2'$, $F_1 - F_2F_3{}^{-1}F_2'$ since they are, according to the Frobenius formula for inversion of block matrices [8], diagonal blocks of the positive-definite matrices Q^{-1} and F^{-1}, respectively. It is easy to show now that (9) represents the corresponding Cauchy problem for the Riccati equation connected with (7), and the conditions of existence and uniqueness of the optimal control in the problem (7) where $\bar{x}_2$ acts as the control, both hold. We know from (9) that the optimal control in (7) has the form $\bar{x}_2 = -Q_3{}^{-1}[Q_2' + A_2'\bar{K}_1]\bar{x}_1$ and this yields

$$\bar{x}_1^{\bullet} = (A_1 - A_2Q_3{}^{-1}A_2'\bar{K}_1 - A_2Q_3{}^{-1}Q_2)\bar{x}_1, \quad \bar{x}_1(t_0) = x_1{}^0$$

Since $v = -R^{-1}B'Kx$ we have, in place of (3),

$$\begin{aligned}
& x_1^{\bullet} = A_1(t)x_1 + A_2(t)t_2, \quad x_1(t_0) = x_1{}^0 \\
& Ex_2^{\bullet} = \varepsilon A_3(t)x_1 + \varepsilon A_4(t)x_2 - R^{-1}(t)(K_2'x_1 + K_3x_2), \\
& x_2(t_0) = x_2{}^0
\end{aligned} \tag{10}$$

Applying now the theorem due to A. N. Tikhonov [1] on the segment $[t_0, T_1]$ (the condition of positive stability of the root $\bar{x}_2^{\circ}(t) = -\bar{K}_3^{-1}\bar{K}_2'\bar{x}_1^0(t)$ holds since $\bar{K}_3(t)$ is a positive-definite matrix) we find that $\lim x_1(t, \varepsilon) = \bar{x}_1^{\circ}(t)$ uniformly in $t \in [T_0, T_1] \subset [t_0, T]$ where $\bar{x}_1^0(t)$, $\bar{x}_2^0(t)$ is the solution of the problem obtained from (10) when $\varepsilon = 0$. Further we have $\bar{K}_3^{-1}\bar{K}_2' = \bar{K}_3^{-1} R\bar{K}_3^{-1}(Q_2' + A_2\bar{K}_1) = \bar{K}_3^{-1}R\bar{K}_3^{-1}Q_2' + \bar{K}_3^{-1}R\bar{K}_3^{-1}A_2'\bar{K}_1' = Q_3^{-1}Q_2' + Q_3^{-1}A_2'\bar{K}_1$, and this makes it clear that $\bar{x}_1^{\circ}(t) = \bar{x}_1(t)$, $\bar{x}_2^{\circ}(t) = \bar{x}_2(t)$, thus proving the first two assertions of the theorem. The third assertion follows from the previous two. Finally, making use of the expression for the minimum value of the functional we obtain $\lim\limits_{\varepsilon\to 0} I_\varepsilon^* = I^*$.

Corollary. $\lim\limits_{\varepsilon\to 0} p_1(t, \varepsilon) = \bar{p}_1(t)$ uniformly in $t \in [t_0, T_1]$, $\lim\limits_{\varepsilon\to 0} p_2(t, \varepsilon) = \bar{p}_2(t) \equiv 0$ uniformly in $t \in [T_0, T_1]$ where $\bar{p}_1(t)$ is the conjugate variable of the Pontriagin maximum principle for the problem (7).

The proof follows from (6), the theorem and the properties of $K_i(t, \varepsilon)$, $i = 1, 2, 3$.

Notes 1°. Making the substitution $v = \varepsilon u$ in (1),(2), we obtain a problem which can be assumed regularized with respect to the problem with a singular optimal control (see below). Having obtained the asymptotics of the solution of the regularized problem in terms of the regularization parameter, we can find the asymptotics of the solution of the corresponding problem with a large gain coefficient accompanying the control (the trajectories coincide at $v = \varepsilon u$). Therefore, using the arguments employed in [4] where the asymptotics of the solution of the regularized problem is constructed using the singular perturbation methods, we find that the optimal control $v(t, \varepsilon)$ in the problem (1),(2) has, under the condition that F is a positive-semidefinite matrix and $F_2 = F_3 = 0$, and provided that the matrix elements in the problem (1),(2) are sufficiently smooth in t, the following asymptotic representation:

$$v(t, \varepsilon) = \sum_{k=0}^{N}\left(\prod\nolimits_k^{\circ} v\left(\frac{t-t_0}{\varepsilon}\right) + \varepsilon v_k(t) + \varepsilon \prod\nolimits_k^{1} v\left(\frac{t-T}{\varepsilon}\right)\right)\varepsilon^k + O(\varepsilon^{N+1})$$

where the terms of the boundary layer series $\Pi_k^i(\tau_i)$, $i = 0, 1$ tend exponentially to zero as $|\tau_i| \to \infty$, i.e. the principal part of the optimal control is concentrated in the boundary layer near t_0.

In the present case (F is a positive-definite matrix) the principal part of the control lies in the boundary layer near t_0 and T, and the explanation of its structure becomes more complex.

2°. Let us make the substitution $v = \varepsilon u$ in (1.),(3) and put $\varepsilon = 0$ in the resulting problem. This yields

$$\min\left\{I(u) = \frac{1}{2}\left\|\begin{matrix}x_1(T)\\ x_2(T)\end{matrix}\right\|' F \left\|\begin{matrix}x_1(T)\\ x_2(T)\end{matrix}\right\| + \frac{1}{2}\int_{t_0}^{T}\left\|\begin{matrix}x_1\\ x_2\end{matrix}\right\|' Q \left\|\begin{matrix}x_1\\ x_2\end{matrix}\right\| dt\right\} \tag{11}$$

$$\dot{x}_1 = A_1(t)x_1 + A_2(t)x_2, \quad x_1(t_0) = x_1^{\circ}$$
$$x_2^{\cdot} = A_3(t)x_1 + A_4(t)x_2 + u, \quad x_2(t_0) = x_2^{\circ}$$

It can be shown that the optimal control is singular [9] and is determined from the condition

$$H_u = p_2 = 0$$
$$(H = p_1' (A_1x_1 + A_2x_2) + p_2' (A_3x_1 + A_4x_2 + u) - {}^1/_2 x'Qx$$
$$H_{uu} \equiv 0, \ p_2 = \bar{p}_2)$$

Minimizing the term of the functional in (11) appearing outside the integral sign we obtain $x_2 (T) = -F_3^{-1}F_2'x_1 (T)$, i.e. the optimal control is, in general, also impulsive at the point $t = T$ [10, 11]. Substituting $x_2 (T)$ into the functional of the problem (11) we obtain a functional just as in (7), i.e. the jump in the value of the functional can be obtained as the result of minimizing the term outside the integral over the finite values of the rapid variables.

3°. As in [4], the essential condition of the present work is, that the matrix

$$B'QB = \left\|\begin{matrix}0\\E\end{matrix}\right\|' Q \left\|\begin{matrix}0\\E\end{matrix}\right\| = Q_3$$

is positive-definite. In more general cases when this condition no longer holds, the control x_2 in the problem of reduced dimension becomes itself singular and further splitting of the problem becomes necessary [10, 12].

REFERENCES

1. Vasil'eva, A.B. and BUTUZOV, V.F., Asymptotic Expansions of Solutions of Singularly-Perturbed Equations. Moscow, "Nauka", 1973.
2. Gerashchenko, E.I. and Gerashchenko, S.M., Method of Separating the Motions and Optimization of Nonlinear Systems. Moscow, "Nauka", 1975.
3. Kelly, H.J., Aircraft maneuver optimization by reduced-order approximation. In: Control and Dynamic Systems. Vol. 10, New York-London, Acad.Press, 1973.
4. O'Malley, R.E.Jr. and Jameson, A., Singular perturbations and singular arcs. pt. 1. IEEE Trans. Automat. Contr., Vol. 20, No. 2, 1975.
5. Dmitriev, M.G., Esipova, V.A. and Chuev, V.I., Limiting passage in a singularly-perturbed problem of optimal control. Differentsial'nye uravneniia i ikh prilozheniia. No. 2, Izd. Dnepropetovsk Univ., 1973.
6. Glizer, V.Ia. and Dmitriev, M.G., On continuity of solution of the problem of a regulator analytic construction in singular perturbations. PMM, Vol. 41, No. 3, 1976.
7. Glizer, V.Ia. and Dmitriev, M.G., Singular perturbations in the linear problem of optimal control with a quadratic functional. Dokl. Akad. Nauk SSSR, Vol. 225, No. 5, 1975.
8. Gantmakher, F.R., Theory of Matrices, (English translation), Chelsea, New York, 1959.
9. Bryson, A.E. and Ho Yu-Chi., Applied Optimal Control. Waltham, Mass. Blaisdell Pub. Co., 1969.
10. Ho Yu-Chi., Linear stochastic singular control problems. J. Optimizat. Theory and Appl., Vol. 9, No. 1, 1972.
11. Zavalishchin, S.T. and Sesekin, A.N., On particular solutions in the problems of optimization of dynamic systems with a quadratic quality criterion. Differentsial'nye uravneniia. Vol. 11, No. 4, 1975.
12. O'Malley, R.E.Jr. and Jameson, A., Singular perturbing and singular arcs. Pt. 2. IEEE Trans. Automat. Constr., Vol. 22, No. 3, 1977.

Translated by L. K.